9/18/25 Lecture 8

Proof by Contradiction

- Assume the negation is true
- Show a contradiction to show the negation is false.

Proof by Cases

- Cases must cover entire domain
- If cases are symmetrical, use "without loss of generality"

Example

x is even $\vee$ y is even $\rightarrow xy$ is even

- This is where you would use "without loss of generality"

For biconditional statement $p \leftrightarrow q$

- you must prove $p \rightarrow q$ and $q \rightarrow p$

Proof strategy

- if statement is $p \rightarrow q$
  - try direct proof
  - try proof by contrapositive
  - try proof by contradiction
- Try to AVOID disjunctions
- To prove a statement is false, show a counter example.

Sets

$A = \{x \in \mathbb{Z}$ x is an integer that is a multiple of $3\} = \{ \ldots -9, -6, -3, 0, 3, 6, 9 \ldots \}$

$B = \{ x \mid x \in \mathbb{Z} \cap x$ is a perfect square. $\} = \{0, 1, 4, 9, 16, 25, \ldots\}$

$C = \{4, 5, 9, 10\}$, $D = \{2, 4, 11, 14\}$, $E = \{3, 6, 9\}$, $F = \{4, 6, 16\}$

$|B| = +\infty$

$|C| = 4$

$U = \mathbb{Z}$

$E \subseteq A \qquad \forall x (x \in E \to x \in A)$

$E \subset A$

If $E \subseteq A$ and $|E| < |A|$ then E is a proper subset of A.

Set Operations.

- Intersection

  $A \cap B = \{x : x \in A \cap x \in B\}$

- Union

  $A \cup B = \{x : x \in A \vee x \in B\}$

- Difference

  $A - B = \{x : x \in A \cap x \notin B\}$

- Complement

  $\bar{A} = \{x \in U : x \in A\}$

Power Sets

$A = \{1, 2\}$, $P(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$

$\emptyset = \{\} \neq \{\emptyset\}$

Proof Example

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Must prove (1) and (2)

you are proving that both sets are subsets of the other

(1) $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

$x \in A \cap (B \cup C)$

$x \in A \land x \in (B \cup C)$

$x \in A \land (x \in B \lor x \in C)$

$(x \in A \land x \in B) \lor (x \in A \land x \in C)$

$(x \in A \cap B) \lor (x \in A \cap C)$

$x \in (A \cap B) \cup (A \cap C)$

(2) $A \cap (B \cup C) \supseteq (A \cap B) \cup (A \cap C)$

you can basically do (1) backwards

Proof Example

$\overline{A \cup B} = \overline{A} \cap \overline{B}$

1. $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$

   $x \in \overline{A \cup B}$

   $x \notin A \cup B$

   $(x \notin A) \cap (x \notin B)$

   $(x \in \overline{A}) \cap (x \in \overline{B})$

   $x \in \overline{A} \cap \overline{B}$

2. $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$

   prove this direction as well to show equality

Cartesian Products

Let $A = \{1, 2\}$, Let $B = \{1, 3\}$

$A \times B = \{(1,1), (1,3), (2,1), (2,3)\}$

If you are doing $A \times B \times C$ and $|A| = 4$, $|B| = 1$, $|C| = 5$

- Then $|A \times B \times C| = |A| \cdot |B| \cdot |C| = 20$

- $\emptyset \times A = \emptyset$
- $|A^k| = |A|^k$
- $A^k \times A^j = A^{k+j}$
- $A^0 = \{\emptyset\}$